Hitchin. Generalized Geometry.

reference : Hitchin IMS lectures in 2010

§ Doubling

Linear alg. $V \simeq \mathbb{R}^n \rightsquigarrow V \oplus V^* \simeq \mathbb{R}^{n,n}$

$GL(n,\mathbb{R}) \subset SO(n,n)$

manifold T_M $\qquad T_M \oplus T_M^*$

$[\ ,\ ] \rightsquigarrow$ Courant bracket

$\mathrm{Diff}\,M \leqslant \mathrm{Diff}\,M \ltimes \Omega^2_{cl}$

B-fields

$\Lambda^\bullet T_M = \$$!

generalized cpx manifold

(include cpx. mfd & sympl. mfd)

generalized cpx. submfd.

(include cpx. submfd. + holo. line bdl. & coisotropic A-brane ($\geqslant$ Lagr. + flat $U(1)$-bdl))

generalized Kähler manifold

§1. Basic $\qquad M^n$

From T to $T\oplus T^* \ni X+\xi$

$(X+\xi, X+\xi) = \iota_X \xi$ $\qquad$ Signature (n,n).

Self-adjoint endomorphism, $\quad \mathfrak{o}(n,n)$

$$\begin{pmatrix} A & \beta \\ B & -A^t \end{pmatrix} : \begin{matrix} T \\ \oplus \\ T^* \end{matrix} \longrightarrow \begin{matrix} T \\ \oplus \\ T^* \end{matrix}$$

eg. $\begin{pmatrix} 0 & 0 \\ B & 0 \end{pmatrix}$, need

$(B(X_1+\xi_1), X_2+\xi_2) = (B(X_1), X_2) \quad = -(X_1, B(X_2))$

$\Rightarrow$ $B : T \to T^*$ skew. i.e. $B \in \Lambda^2 T^*$

$\begin{pmatrix} 0 & 0 \\ B & 0 \end{pmatrix}^2 = 0 \Rightarrow$ orthogonal auto. $\quad X+\xi \mapsto X+\xi+\iota_X B$

$e^{\begin{pmatrix} 0 & 0 \\ B & 0 \end{pmatrix}} = I + \begin{pmatrix} 0 & 0 \\ B & 0 \end{pmatrix}$ $\qquad$ B-field action.

- Lie bracket $\rightsquigarrow$ Courant bracket

$$[X+\xi, Y+\eta]$$
$$= \underbrace{[X,Y]}_{T} + \underbrace{\mathcal{L}_X \eta - \mathcal{L}_Y \xi - \frac{1}{2} d(\iota_X \eta - \iota_Y \xi)}_{T^*}$$

$\nexists$ Jacobi identity, i.e. not Lie alg.

Prop: $[\ ,\ ]$ is preserved by closed B-fields.

Pf. $[X+\xi+\iota_X B, Y+\eta+\iota_Y B]$

$= [X+\xi, Y+\eta] + \mathcal{L}_X(\iota_Y B) - \frac{1}{2} d(\iota_X \iota_Y B)$
$\quad - \cancel{\mathcal{L}_Y}(\iota_X B) + \frac{1}{2} d(\iota_Y \iota_X B)$

$d\,\iota_Y \iota_X B = \mathcal{L}_Y(\cancel{\iota_X}B) - \iota_Y d(\iota_X B)$

$= [X+\xi, Y+\eta] + \iota_{[X,Y]} B + \underbrace{\iota_Y \mathcal{L}_X B - \iota_Y d\, \iota_X B}_{\iota_Y \iota_X \cancel{dB}}$ $\quad (\because B \text{ closed})$

• $\mathrm{Diff}\,M \ltimes \Omega^2_{cl}$

$$X + \xi \in \Gamma(T \oplus T^*)$$
$$\mapsto X - d\xi \in \mathrm{Lie}(\mathrm{Diff}(M) \ltimes \Omega^2_{cl}) =: \mathfrak{g}$$

acts on $\Gamma(T \oplus T^*)$:

$$Y + \eta \xrightarrow{X - d\xi} \mathcal{L}_X(Y+\eta) - \iota_Y d\xi =: u\,v \qquad \begin{matrix} u = X+\xi \\ v = Y+\eta \end{matrix}$$

Ex: Courant bracket $= \frac{1}{2}(uv - vu)$

Ex: $\frac{1}{2}(uv + vu) = \frac{1}{2}(\mathcal{L}_X\eta - \iota_Y d\xi + \mathcal{L}_Y\xi - \iota_X d\eta)$
$$= \frac{1}{2} d(\iota_X\eta + \iota_Y\xi)$$
$$= d\,(u, v).$$

Prop: $u(vw) = (uv)w + v(uw)$. i.e. derviat^{n}

Pf. $u = X + \xi \xrightarrow{\text{write}} \tilde{u} = X - d\xi$

$u(vw) - v(uw) = \tilde{u}\tilde{v}(w) - \tilde{v}\tilde{u}(w) = [\tilde{u}, \tilde{v}](w)$ ← Lie bracket in $\mathfrak{g}$

$$[\tilde{u}, \tilde{v}] = [X, Y] - (\mathcal{L}_X d\eta - \mathcal{L}_Y d\xi)$$

$$uv = [X, Y] + \mathcal{L}_X\eta - \iota_Y d\xi \qquad (d\,\iota_Y d\xi = \mathcal{L}_Y d\xi - \iota_Y \cancel{d^2\xi})$$

acts as $[X, Y] - d(\mathcal{L}_X\eta - \iota_Y d\xi)$

Cor. For Courant bracket, (Jacobi id. up to exact).

$$[[u, v], w] + \text{cyclic} = \frac{1}{3} d\big(([u, v], w) + \text{cyclic}\big)$$

Pf: LHS: $\frac{1}{4}\begin{pmatrix} (\underbrace{uv}_{} - vu)w - w(uv - vu) \\ +(vw - wv)u - u(vw - wv) \\ +(wu - uw)v - v(wu - \underbrace{uw}_{\text{pair up}}) \end{pmatrix} \rightsquigarrow (-1)\times$ sum of right hand column.

$\qquad\qquad\qquad\quad l \qquad\qquad\qquad\quad r$

$$l + r = -r \quad \& \quad l - r = -3r = 3(l + r)$$

$$l + r = \frac{1}{3}(l - r)\frac{1}{4}\left\{\begin{matrix} \underbrace{(uv - vu)w + w(uv - vu)}_{\to\, 4\,d([u,v],w)} \\ + \\ + \end{matrix}\right\}$$

#

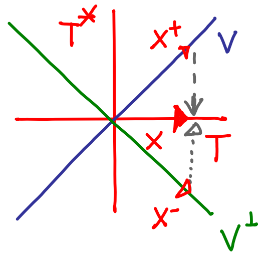
§2 Riemannian geometry.

$$(M,\ g = g_{ij}\, dx_i \otimes dx_j) \rightsquigarrow g : T \longrightarrow T^*$$
$$\frac{\partial}{\partial x_i} \mapsto g_{ij}\, dx_j$$

(not nec: $g_{ij} = g_{ji}$)

$\rightsquigarrow$ $V :=$ graph of $\underbrace{g \subset T \oplus T^*}_{\text{subbdl}}$

$V^{\perp} :=$ graph of $\underbrace{(-g) \subset T \oplus T^*}_{\text{subbdl}}$



Prop: $X \in \Gamma(T)$ and $v \in \Gamma(V)$

$$\nabla_X v := \underbrace{\Pi_V}_{\text{proj. to } V} [X^-, v]_{\text{courant}}$$

is a connection on V

preserves inner product induced on V.

reason: Properties of Courant bracket

$[u, fv] = f[u,v] + (Xf)v - (u,v)\, df.$

$X(v,w) = ([u,v] + d(u,v), w) + (v, [u,w] + d(u,w))$

In fact, only need symmetric part of g to be pos. def.

$\operatorname{Tor} \nabla = d(\operatorname{skew}(g)).$

Realization of Christoffel symbol:

$$\left(T \ni \frac{\partial}{\partial x_i} \leftrightarrow \frac{\partial}{\partial x_i} - g_{il}\, dx_l \in V^{\perp},\ \frac{\partial}{\partial x_i} + g_{il}\, dx_l \in V\right)$$

$$\nabla_{\underbrace{\frac{\partial}{\partial x_i}}_{X}} \Big(\underbrace{\frac{\partial}{\partial x_j} + g_{jk}\, dx_k}_{v}\Big) = \Pi_V [X^-, v]_{\text{courant}}$$

$$= \Pi_V\left(\frac{\partial g_{jl}}{\partial x_i} dx_l + \frac{\partial g_{il}}{\partial x_j} dx_l - \frac{1}{2}\frac{\partial}{\partial x_l}(g_{ji} + g_{ij})\, dx_l\right)$$

$$= \underbrace{\frac{1}{2} g^{lk}\left(\frac{\partial g_{jl}}{\partial x_i} + \frac{\partial g_{il}}{\partial x^j} - \frac{\partial g_{ij}}{\partial x_l}\right)}_{\Gamma^k_{ij}} \left(g_{ui}\, dx_i + \frac{\partial}{\partial x_u}\right) \quad \begin{pmatrix}\text{assume} \\ g_{ij} = g_{ji}\end{pmatrix}$$

§ Spinor $O(n,n)$ on $T\oplus T^*$, $(\ ,\ )$

$\rightsquigarrow$ Spinor $\$ = \Lambda^\bullet T^*$ $(\otimes (\Lambda^{TOP} T^*)^{1/2})$

Clifford action $(T+T^*) \times \Lambda^\bullet T^* \longrightarrow \Lambda^\bullet T^*$

$$(X+\xi)\cdot\varphi := \iota_X \varphi + \xi\wedge\varphi$$

$$\left(\Longrightarrow (X+\xi)^2\cdot\varphi = \cdots\cdots = (\iota_X\xi)\varphi = (X+\xi, X+\xi)\varphi\right)$$

B-field action on $\$ \ni \varphi$, $B\in\Lambda^2 T^*$

$$\left(B\wedge(\iota_X+\xi)\varphi - (\iota_X+\xi\wedge)B\wedge\varphi = -(\iota_X B)\varphi - B\wedge\cancel{\iota_X\varphi} + \cancel{B\wedge\iota_X\varphi}\right)$$

$$\rightsquigarrow \quad \varphi \mapsto e^{-B\wedge}\varphi.$$

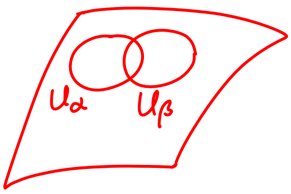
- On manifold. $\mathrm{Diff}(M)\ltimes\Omega^2_{cl} \curvearrowright \Gamma(\$)$

$$(X - d\xi)\cdot\varphi = \mathcal{L}_X\varphi + d\xi\wedge\varphi$$
$$= d\,(X+\xi)\cdot\varphi + (X+\xi)\cdot d\varphi$$

"Cartan" formula.

§ Twisted structures

$T\oplus T^*$, $(\ ,\ )$, $[\ ,\ ]$ preserved by closed B-fields.



$$B_{\alpha\beta}\in\Omega^2_{cl}(U_\alpha\cap U_\beta)$$

$$B_{\alpha\beta} + B_{\beta\gamma} + B_{\gamma\alpha} = 0 \text{ on } U_{\alpha\beta\gamma}$$

$$(T\oplus T^*)|_{U_\alpha} \cong (T\oplus T^*)|_{U_\beta} \text{ on } U_{\alpha\beta}$$

via $X+\xi \longmapsto X+\xi+\iota_X B_{\alpha\beta}$

$\rightsquigarrow$ vector bundle E, locally modelled on $T\oplus T^*$

indeed, an ext$^{\underline{n}}$. $0\longrightarrow T^*\longrightarrow E\longrightarrow T\longrightarrow 0$

$\rightsquigarrow$ $(E, (\ ,\ ), [\ ,\ ])$, exact Courant algebroid

- Consider $0\longrightarrow\Omega^2_{cl}\xrightarrow{d}\Omega^3\longrightarrow\Omega^3_{cl}\longrightarrow 0$, ex. seq. of shf.

$$[B_{\alpha\beta}]\in\check{H}^1(\Omega^2_{cl}) = H^0(\Omega^3_{cl})/dH^0(\Omega^2) = H^3_{dR}(M,\mathbb{R})$$

(non-canon) $(T+T^*, \langle\ \rangle, [\ ]_H := [\ ] + \iota_X\iota_Y H)$ $\exists H \in \Omega^3_{cl}$

- Twisted de Rham cohomology

$(E, (\ ,\ )) \rightsquigarrow$ Spinor bundle S:

$$\Lambda^\bullet T^*|_{\mathcal{U}_\alpha} \cong \Lambda^\bullet T^*|_{\mathcal{U}_\beta} \text{ on } \mathcal{U}_{\alpha\beta}$$

$$\text{via} \quad \varphi \longmapsto e^{-B_{\alpha\beta}}\varphi$$

$$d\varphi = e^{B_{\alpha\beta}}\, d\,(e^{-B_{\alpha\beta}}\varphi) \quad (\because dB_{\alpha\beta} = 0)$$

$$\Rightarrow d : \Gamma(S^{ev}) \longrightarrow \Gamma(S^{odd})$$

$$\rightsquigarrow H^\bullet_{[B]}(M, \mathbb{R}) := \left.\frac{\operatorname{Ker} d}{\operatorname{Im} d}\right|_{\Gamma(S^\bullet)}$$

Choose any isotropic splitting

$$\begin{tikzcd} 0 \arrow[r] & T^* \arrow[r] & E \arrow[r] & T \arrow[r] & 0 \arrow[l, bend right] \end{tikzcd}$$

$$\rightsquigarrow F_\alpha \in \Omega^2(\mathcal{U}_\alpha) \quad \text{s.t. } F_\beta - F_\alpha = B_{\alpha\beta}$$

$$\rightsquigarrow H := dF_\alpha = dF_\beta \quad (\because dB_{\alpha\beta} = 0)$$

$$\text{i.e. } H \in \Omega^3_{cl}(M)$$

a section of S

$$\longleftrightarrow \varphi_\alpha \in \Omega^\bullet(\mathcal{U}_\alpha) \text{ s.t. } \varphi_\alpha = e^{-B_{\alpha\beta}}\varphi_\beta \text{ on } \mathcal{U}_{\alpha\beta}$$

$$\rightsquigarrow \psi := e^{-F_\alpha}\varphi_\alpha = e^{-F_\beta}\varphi_\beta \in \Omega^\bullet(M)$$

$$d\varphi_\alpha = 0 \quad \forall \alpha \iff d\psi = -(dF_\alpha)\psi$$

$$\iff (d + H)\psi = 0$$

($\sim$ gerbes)

Given twisted structure,

$$0 \longrightarrow T^* \longrightarrow E \longrightarrow T \longrightarrow 0 \quad \text{w/ } (\ ,\ ),\ [\ ,\ ]$$

Def^{n}. A generalized metric in E is a subbdl $V \subset E$ of rank n s.t. $(\ ,\ )|_V > 0$.

i.e. reduction $O(n,n) \geq O(n)^2$

Locally, V is a graph of $h_\alpha : T \to T^*$

$$h_\alpha = g_\alpha + F_\alpha \in \mathrm{Sym}^2 T^* \oplus \Lambda^2 T^*, \quad \exists\, g_\alpha$$

On $\mathcal{U}_\alpha \cap \mathcal{U}_\beta$, $g_\alpha = g_\beta$ $\rightsquigarrow$ global metric

$$F_\beta - F_\alpha = B_{\alpha\beta}.$$

$$\because \left.\begin{matrix} (\ \ )|_V > 0 \\ (\ \ )|_{T^*} = 0 \end{matrix}\right\} \Longrightarrow \begin{tikzcd} E \arrow[r] & T \arrow[r] & 0 \\ V \arrow[u, phantom, "\cup"] \arrow[ur, "\cong"'] & & \end{tikzcd}$$

$$\rightsquigarrow \quad \nabla_X v = \pi_V [X^-, v].$$

$$\left[\frac{\partial}{\partial x_i} - g_{ik}\, dx_k + F_{ik}\, dx_k,\ \frac{\partial}{\partial x_j} + g_{jl}\, dx_l + F_{jl}\, dx_l\right]$$

$$= \text{Levi-Civita} + \frac{\partial F_{jl}}{\partial x_i} dx_l - \frac{\partial F_{ik}}{\partial x_j} dx_k - \frac{1}{2} d(F_{ji} - F_{ij})$$

$$= \left(\frac{\partial F_{jl}}{\partial x_i} - \frac{\partial F_{il}}{\partial x_j} - \frac{\partial F_{ji}}{\partial x_l}\right) dx_l$$

skew-torsion $\quad dF_\alpha = H \in \Omega^3(M)$

$\rightsquigarrow$ Riemannian metric w/ skew-torsion.

Eg 1. G Lie group w/ bi-inv. metric

Define $\nabla_X Y = 0$ for left inv. vector fields

$\Longrightarrow$ flat metric w/ skew-torsion

Eg 2. $\forall$ Hermitian mfd., $\exists!$ (Bismut) connection

$$\nabla g = 0 = \nabla J, \quad \mathrm{Tor}(\nabla) = d^c\omega = J d\omega =: H$$

$$dH = 0 \iff dd^c\omega = 0$$

i.e. Strong Kähler w/ torsion (SKT) metric.

gen. metric $\Leftrightarrow$ usual metric + isotropic splitting $V_+ = \mathrm{Im}(s+g)$.

$\rightsquigarrow$ closed 3-form $H = \langle s, [s,s]\rangle$.

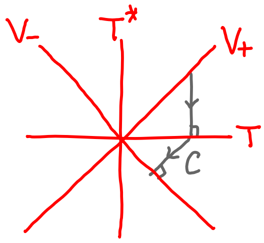
- Generalized connection $D : \Gamma(E) \longrightarrow \Gamma(E^* \otimes E)$

  s.t. Leibniz & compat. w/ $\langle\ \rangle$.

  Torsion $T_D \in \Lambda^3 E$

  $$T_D(e_1, e_2, e_3) = \langle D_{e_1} e_2 - D_{e_2} e_1 - [e_1, e_2], e_3 \rangle + \langle D_{e_3} e_1, e_2 \rangle$$

- Gualtieri-Bismut connection



  Given gen. metric V_+ $\quad (V_\pm \simeq T)$

  $\rightsquigarrow$ (1) $C : E \longrightarrow E$ via projection

  $$C(V_\pm) = V_\mp$$

  (2) Connection $D^B_e f = [e_-, f_+]_+ + [e_+, f_-]_- + [Ce_-, f_-]_- + [Ce_+, f_+]_+$

  D^B preserves $V_\pm$, torsion T_{D^B} of D^B is skew-symm.

  Project to T $\rightsquigarrow$ $\nabla^\pm = \nabla^g \pm \frac{1}{2} g^{-1} H$ — 2 metric conn. w/ skew-symm. torsion

- $D^{LC} := D_B - \frac{1}{3} T_{D^B}$

  $\rightsquigarrow$ $D^{LC}_\pm : V_+ \longrightarrow V_+ \otimes V_\pm^*$ by restriction

  $\rightsquigarrow$ $\not{D}^{LC}_+ : \not{S}_+(V_+) \longrightarrow \not{S}_-(V_+)$ assuming $\dim M = 2n$.

§ Killing spinors

Recall : Def. $\eta \in \Gamma(\$_M)$ Killing Spinor if

$\exists\ A \in \Gamma(\mathrm{End}\, T_M)$ s.t. $\nabla_X \eta = A(X) \cdot \eta \quad \forall\ X$

- $A = \lambda I \Rightarrow Rc \equiv C$
- $A = 0$, η pure $\Rightarrow$ special holonomy

Def. $\eta \in \$_+(V_+)$ Killing spinor if $D_+^{LC}\eta = 0 = D_-^{LC}\eta$

Prop (Fernandez-Rubio-Tipler) $\eta \neq 0$ pure Killing spinor

$\Rightarrow \quad H = 0$ & g CY metric.

- $[\hat{H}] \in H^3(P, \mathbb{R})^G$ for $G \to P \to M$

$\rightsquigarrow \quad 0 \to T^*P \longrightarrow \hat{E} \longrightarrow TP \longrightarrow 0$

equivar. exact Courant algebroid $/P$

"IF" (i) $\exists\, \rho$

$$\begin{tikzcd} & \Gamma(\hat{E}) \arrow[d] \\ \mathfrak{g} \arrow[r] \arrow[ur, dashed, "\rho"] & \Gamma(TP) \end{tikzcd}$$

(ii) $c(z_1, z_2) \triangleq \langle \rho(z_1), \rho(z_2) \rangle$ non-degenerate

$\Rightarrow$ descend

$$E \triangleq \frac{\rho(\mathfrak{g})^\perp}{\rho(\mathfrak{g}) \cap \rho(\mathfrak{g})^\perp} / G \underset{\text{non-canon.}}{\simeq} T \oplus \mathrm{ad}P \oplus T^*$$

$$\downarrow$$

$$M = P/G$$

Choose any G-equivar. isotropic splitting of $\hat{E}$

$\rightsquigarrow$ connection A on P

s.t. $\rho(z) = Y_z + c(z, A\cdot)$

$$\hat{H} = p^* H - CS(A)$$

$$d\hat{H} = 0 \Rightarrow dH = c(F_A^2) \Rightarrow p_1(P) = 0$$

- In physics, $P = P_{fr} \times P_K$ w/ $K \leq E_8 \times E_8$
- $\exists \langle \ \rangle$ & $[\ \ ]$ on E

Theorem (Fernandez-Rubio-Tipler)

(V, η_+) pure Killing spinor w/ signature $2n$

$\Leftrightarrow$ Strominger system (ω, A) on CY^n (X, Ω)

w/ $H = d^c\omega$, $[\hat{H}] = [p^* d^c\omega - CS(A)]$

$$\begin{cases} \Lambda F_A = 0 = F_A^{0,2} \\ d\,\omega^{n-1} = 0 \\ \partial\bar{\partial}\omega = c(F_A^2) \end{cases}$$

Here, $|\Omega| = 1$ is assumed, otherwise need conformal generalized geometry.

§ Generalized complex structure

Recall usual complex structure.

(Linear) $J: T \to T$ w/ $J^2 = -1$

$\rightsquigarrow (+i)$-eigenspace in $T \otimes \mathbb{C} \rightsquigarrow T^{1,0}$

(Integrability) $[T^{1,0}, T^{1,0}] \subset T^{1,0}$ (for sections)

Def: Generalized complex structure is

$$\mathcal{J}: T \oplus T^* \longrightarrow T \oplus T^* \quad \text{w/} \quad \mathcal{J}^2 = -1$$

$$(\mathcal{J}u, v) = -(u, \mathcal{J}v) \quad \text{(i.e. } U(n,n)\text{-str.)}$$

$\rightsquigarrow (+i)$-eigenspace in $(T \oplus T^*) \otimes \mathbb{C} \rightsquigarrow E^{1,0}$

(Integrability) $[E^{1,0}, E^{1,0}]_{\text{Courant}} \subset E^{1,0}$

Claim: $E^{1,0}$ is max. isotropic in $(T \oplus T^*) \otimes \mathbb{C}$

$$\left[(u,v) \overset{u \in E^{10}}{=\!=} -i(\mathcal{J}u, v) \overset{\mathcal{J}\text{ortho.}}{=\!=} +i(u, \mathcal{J}v) \overset{v \in E^{10}}{=\!=} -(u,v) \right.$$

- $u, v \in E^{10} \Rightarrow [u, fv] = f[u,v] + (Xf)v - \cancel{(u,v)}\, df$

  $\rightsquigarrow$ tensorial.

- real $GL(2n, \mathbb{R}) \supset GL(n, \mathbb{C})$ cpx.

$$\begin{array}{lccccl} \text{real} & GL(2n,\mathbb{R}) & \supset & GL(n,\mathbb{C}) & & \text{cpx.} \\ & \cap & & \cap & & \\ \text{gen.} & O(2n,2n) & \supset & U(n,n) & & \text{gen. cpx.} \end{array}$$

Eg 1. Complex mfd. $E^{1,0} = \langle \frac{\partial}{\partial \bar{z}_i}, dz_i \rangle$

2. Symplectic mfd. $E^{1,0} = \langle \frac{\partial}{\partial x}, -i\underbrace{\omega_{jk}}_{\text{B-field}} dx_k \rangle$.

$T \subset T \oplus T^*$

$\frac{\partial}{\partial x^j} \mapsto \frac{\partial}{\partial x^j} - i\omega_{jk} dx_k$

closed 2-form preserves [,]

3. Holomorphic Poisson manifold (M, J, σ)

$$\sigma \in H^0_{\bar\partial}(\Lambda^2 T) \rightsquigarrow \sigma : T^* \longrightarrow T$$

$f : M \to \mathbb{C} \rightsquigarrow$ Hamil. v.f. $X_f := \sigma(df)$

Poisson bracket $\{f, g\} := X_f(g) = \sigma(df, dg) = -\{g, f\}$

Integrability : $[X_f, X_g] = \sigma(d\{f, g\})$ Lie bracket

$$E^{1,0} := \langle \frac{\partial}{\partial \bar z} , \ dz - \sigma(dz) \rangle$$ Integrable ✓

i.e. $[dz_i - \sigma(dz_i), dz_j - \sigma(dz_j)] = \sigma d\{z_i, z_j\} - d\{z_i, z_j\}$

(can also be viewed as "B-field" transf. w/ $B = \sigma$)

Eg. M complex surface w/ anti-canon. curve
$\cup\!\!\!\text{I}$
$C = \{\sigma = 0\}$ $\quad \sigma = H^0(K^{-1}) = H^0(\Lambda^2 T)$

integrability is automatic.

- max. isotropic $E^{1,0} \leq (T \oplus T^*)^c$

$\sim$ pure spinor $\varphi \in S^{\mathbb{C}} = \Lambda^\bullet T^* \otimes \mathbb{C}$

(pure $\triangleq$ $\dim \underbrace{\mathrm{Ann}(\varphi)}_{E^{1,0}}$ maximal)

Eg. complex $\varphi = dz_1 \wedge \cdots \wedge dz_n$; symplectic $\varphi = e^{i\omega}$

(Generalized CY $\sim$ $d\varphi = 0$)

Integrability $\Leftrightarrow$ $d\varphi = \theta \cdot \varphi$, $\exists\, \theta \in \Gamma((T \oplus T^*)^c)$
Clifford multi.

$u = X + \xi \in \Gamma(T \oplus T^*) \rightsquigarrow \tilde u = X - d\xi \in \mathrm{Lie}(\mathrm{Diff}(M) \ltimes \Omega^2_{cl})$

- $\tilde u(v) = u v$ $\rightsquigarrow L_u$ Lie derivative wrt $\tilde u$

- $L_u \varphi = d(u \cdot \varphi) + u \cdot (d\varphi)$

Prop. Assume $d\varphi = \theta \cdot \varphi$

$$\left.\begin{array}{l} u\cdot\varphi = 0 \\ v\cdot\varphi = 0 \end{array}\right\} \Longrightarrow [u,v]\cdot\varphi = 0$$

Pf: $0 = L_v(u\cdot\varphi) = (L_v u)\cdot\varphi + \underbrace{u\cdot(L_v\varphi)}_{=0\,?}$

$$u\cdot(L_v\varphi) = u\cdot d(\cancel{v\cdot\varphi}) + u\cdot(v\cdot\underbrace{d\varphi}_{\theta\cdot\varphi})$$
$$= u\cdot(\underline{v\cdot\theta}\cdot\varphi)$$
$$= u\cdot(2(v,\theta) - \theta\cdot v)\cdot\varphi$$
$$= 2(v,\theta)\,\cancel{u\cdot\varphi} - u\cdot\theta\cdot(\cancel{v\cdot\varphi})$$
$$= 0$$

$$2[u,v]\varphi = (L_v u)\varphi - (L_u v)\varphi = 0$$

Note: $f : M \longrightarrow \mathbb{R}$

$\rightsquigarrow \mathcal{J}(df) =: X + \xi \in \Gamma(T \oplus T^*)$

$\rightsquigarrow X - d\xi \in \mathrm{Lie}(\mathrm{Diff}(M) \ltimes \Omega^2_{cl})$

This is infinitesimal symmetry of $\mathcal{J}$.

Eg. Symplectic $\Rightarrow \mathcal{J}(df) = X_f$ Hamil. v.f.

Eg. Complex $\Rightarrow \mathcal{J}(df) = J(df)$

$\rightsquigarrow dJdf = i\partial\bar{\partial}f$ closed (1,1) real form

§ The $\bar\partial$-complex

$$(T\oplus T^*)^c = E^{1,0}\oplus E^{0,1} \qquad (E^{0,1})^* = E^{1,0}$$

$$\bar\partial_J = \pi^{0,1}\circ d : \Gamma(\mathbb{C}_M) \longrightarrow \Gamma(E^{0,1})$$

Eg. complex symplectic holo. Poisson.

$$\bar\partial_J f = \bar\partial f \ ; \ \tfrac{1}{2}(2X_f + df) \ ; \ \bar\partial f + \sigma(\partial f) - \bar\sigma(\bar\partial f)$$

Extend to $\bar\partial_J : \Gamma(E^{0,1}) \longrightarrow \Gamma(\Lambda^2 E^{0,1})$

via $2(\bar\partial_J \alpha)(u,v) = X(\alpha(v)) - Y(\alpha(u)) + \alpha([u,v])$

where $u = X+\xi,\ v = Y+\eta \in \Gamma(E^{1,0})$

$(\bar\partial_J)^2 = 0$ (okay because $E^{0,1}$ isotropic)

(Jacobi identity holds for $\Gamma(E^{0,1})$)

$\rightsquigarrow \bar\partial_J : C^\infty(\Lambda^p E^{0,1}) \longrightarrow C^\infty(\Lambda^{p+1} E^{0,1})$, w/ $\bar\partial_J^2 = 0$

Eg. For complex case, $E^{0,1} = \bar{T}^* \oplus T$

and $\bar\partial_J = \bar\partial$ on $\Omega^{0,*}(\Lambda^\bullet T)$.

- Twisting on a complex manifold

  replace $T \oplus T^*$ by E via $H \in \Omega^3_{cl}$; patch by $B_{\alpha\beta}$

$B^{1,1}_{\alpha\beta}$ - transform (~ holom. bundle)

$$\rightsquigarrow \bar\partial_J + H^{1,2} : \Omega^{0,l}(\Lambda^m T) \longrightarrow \Omega^{0,l+1}(\Lambda^m T) + \Omega^{0,l+2}(\Lambda^{m-1} T)$$

This is elliptic complex.

Note: $J \in \mathcal{M}_{cpx} \subset \mathcal{M}_{gen.\,cpx}$.

$$T_J \mathcal{M}_{gen.\,cpx} = H^0(\Lambda^2 T) \oplus H^1(T) \oplus H^2(\mathcal{O})$$

(same as deformations of $D^b(M)$!)

Obstructions in $H^0(\Lambda^3 T) \oplus H^2(T) \oplus H^3(\mathcal{O})$
$\oplus\ H^1(\Lambda^2 T) \oplus \ \ldots\ldots$

§ Generalized Complex submanifolds.

submfd. $C \subset M$

$$\rightsquigarrow \underbrace{T_C \oplus N^*_{C/M}}_{\tau_C} \subseteq (T_M \oplus T^*_M)|_C$$

τ_C generalized tangent bdl.

Eg. $C \subseteq (M, J)$ cpx. mfd.

C cpx. submfd $\Longleftrightarrow$ τ_C is $\mathcal{J} = \begin{pmatrix} J & \\ & -J^* \end{pmatrix}$-inv.

Eg. $C \subseteq (M, \omega)$ sympl. mfd.

C Lagr. submfd $\Longleftrightarrow$ τ_C is $\mathcal{J} = \begin{pmatrix} & \omega^{-1} \\ \omega & \end{pmatrix}$-inv. (Ex.)

Problem w/ this as def$^{\underline{n}}$: NOT respect B-field transformations.

Def: Generalized complex submfd. of $(M, \mathcal{J})$ is a pair $C \subset M$ & $F \in \Omega^2(C)$ s.t. $dF = H|_C$.

$\tau_C^F \triangleq \{ X + \xi \in T_C \oplus T_M^*|_C : \iota_X F = \xi|_C \}$ is $\mathcal{J}$-inv.

(a real max. isotropic subbdl. of $(T_M \oplus T_M^*)|_C$)

Note: $e^B \cdot (C, F) := (C, F + B)$

This action preserves $dF = H|_M$ condition.

Eg. (M, J) complex manifold

(C, F) gen. cpx. submfd. of $(M, \mathcal{J}_J)$

$\Longleftrightarrow$ $C \subset M$ cpx. submfd. and $F \in \Omega^{1,1}(C)$

Eg. (M, ω) symplectic manifold

(C, F) is gen. cpx. submfd. of $(M, \mathcal{J}_\omega)$?

- τ_C^F is $\mathcal{J}_\omega$-stable

$\Longleftrightarrow$ τ_C is stable under

$$e^{-F} \mathcal{J}_\omega e^F = \begin{pmatrix} -\omega^{-1} B & -\omega^{-1} \\ \omega + B\omega^{-1} B & B\omega^{-1} \end{pmatrix}$$

$\Longleftrightarrow$ $\omega^{-1}(N_{C/M}^*) \subset T_C$ (i.e. coisotropic)

$\omega^{-1}(\iota_{T_C} F) \subset T_C$ (i.e. F descends to $T_C/T_C^\perp$)

$(\omega + F\omega^{-1}F)(T_C) \subset N_{C/M}^*$ (i.e. $(\omega|_C)^{-1} \circ F$ alm. cpx. str. on $T_C/T_C^\perp$)

$\Longleftrightarrow$ Kapustin coisotropic A-brane !

Def: $\mathbb{C}^r \longrightarrow V \longrightarrow (M, \mathcal{J})$ ← gen. cpx. mfd.

Generalized holomorphic bundle is

$$\bar{D} : \Gamma(V) \longrightarrow \Gamma(V \otimes E^{0,1})$$

s.t. $\bar{D}(fs) = s \otimes \bar{\partial}_J f + f \bar{D} s$

and $(\bar{D})^2 = 0$.

Example: 1. Any gen. cx. str. has canonical bundle $K \subset (\Lambda^* T^*)^c$ as gen. holo. bdl.

$u \in E^{1,0} \iff u \cdot \varphi = 0$ for $\varphi \in (\Lambda^* T^*)^c$

fiber of $K = \mathbb{C} \cdot \varphi$

integrability: $d\varphi = \theta \cdot \varphi$ $\exists! \, \theta \in E^{0,1}$

Eg. 1 Canonical line bundle $K \subset (\Lambda^\bullet T^*)^c$ $\forall$ gen. cpx. str.

$E^{1,0} = \mathrm{Ker}(\varphi \bullet) = 0$ $\exists$ pure spinor $\varphi \in (\Lambda^\bullet T^*)^c$ (unique up to scalar).

integ. $\sim$ $d\varphi = \theta \varphi$ θ

$$K := \mathbb{C}\varphi$$

$$\bar{D}(f\varphi) := (\bar{\partial}_J f + f \theta^{0,1}) \cdot \varphi$$ well-def^d.

$$\bar{D}^2 = 0 \iff \bar{\partial}_J \theta = 0$$

$u, v \in E^{1,0}$, $(\bar{\partial}_J \alpha)(u,v) = X(\alpha, v) - Y(\alpha, u) + \alpha[u,v]$.

Eg. 2. J = ordinary cx. structure.

$$\bar{D} : V \longrightarrow V \otimes (T^{0,1*} \oplus T^{1,0})$$

$$\bar{D}(fs) = s \otimes (\underbrace{\bar{\partial}_J f}_{\text{holo.}}, 0) + f \bar{D} s$$

$$\bar{D} s = (\underbrace{\bar{\partial}_A s}_{\substack{\text{usual holo.} \\ \text{str. for } V}}, \phi s) \qquad \phi \in C^\infty(\mathrm{End} V \otimes T^{1,0})$$

$$0 = \bar{D}^2 s = (\bar{\partial}_A^2 s, (\bar{\partial}_A \phi) s, \phi \wedge \phi \, s)$$

holo. str. on V $\quad$ ϕ is holo $\quad$ $\phi \wedge \phi = 0 \in H^0(\mathrm{End} V \otimes \Lambda^2 T)$

§ Generalized Kähler manifolds

Kähler = cpx + symp. $\rightsquigarrow$ 2 gen. cpx. str.

Def: Generalized Kähler mfd $(M, \underbrace{\mathcal{J}_1, \mathcal{J}_2}_{\text{gen. cpx. str.}})$

s.t. $\mathcal{J}_1 \mathcal{J}_2 = \mathcal{J}_2 \mathcal{J}_1$ and $(\mathcal{J}_1 \mathcal{J}_2 u, u) > 0$

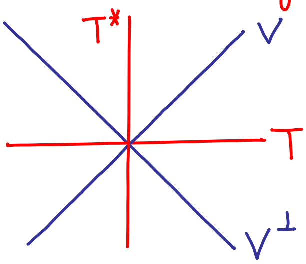
Theorem. M^{2n} Gen. Kähler $\Longrightarrow$

- metric g • integrable Herm. cpx. str. $I^{\pm}$
- $\nabla^{\pm} g = 0 = \nabla^{\pm} I^{\pm}$ and $\mathrm{Tor}(\nabla^{\pm}) = \pm H$
- closed 2-form (equiv. up to B-field action)

(Gate - Hull - Rocek 1984).

$(\mathcal{J}_1 \mathcal{J}_2)^2 = 1 \rightsquigarrow (\pm 1)$- eigenspaces V and $V^{\perp}$.

dim (eigenspace) $= 2n$ $(\because (\mathcal{J}_1\mathcal{J}_2 -, -) > 0)$.



$\rightsquigarrow$ metric w/ skew torsion

$$\begin{cases} \mathcal{J}_i|_{V \simeq T} \rightsquigarrow I^{+} \\ \mathcal{J}_i|_{V^{\perp} \simeq T} \rightsquigarrow I^{-} \end{cases}$$

Thm. (Goto) (M, J, ω) compact Kähler w/ σ holo. Poisson

$\mathcal{J}_1(t)$ family of gen. cpx. str. def^{d} by $t\sigma$, $\mathcal{J}_1(0) = J$

$\Longrightarrow \forall$ small t, $\exists$ gen. Kähler str. $(\mathcal{J}_1(t), \mathcal{J}_2(t))$

s.t. $\mathcal{J}_2(0) = \omega$.

- $[I^{+}, I^{-}] = \mathrm{Re}(\sigma) \in \Gamma(\Lambda^2 T)$